Hypothesis Test & Confidence Interval Intro.

Suppose we know the distribution type of X but not the value of a parameter.

Main Example: Know $X \sim \text{Normal}(\mu, \sigma)$ but don't know μ.

We can pick an unbiased estimator $\hat{\Theta}$ for θ

(so $E[\hat{\Theta}] = \theta$ which means $\hat{\theta} \approx \theta$ and $\hat{\theta} \to \theta$ as $n \to \infty$.)

What if we can't let $n \to \infty$?

How close is $\hat{\theta}$ to θ?

Crazy idea: Think of pdf of X with θ unknown as a joint pdf $f(x;\theta)$ for (X, θ).

To make this an honest joint pdf we need to choose $P(\text{param} = \theta)$ for each θ ⤳ Use uniform dist.

$$P(X = x \ \&\ \text{param} = \theta) = \underbrace{P(X = x \mid \text{param} = \theta)}_{f(x)} \cdot P(\text{param} = \theta)$$

Hypothesis Testing & Confidence Intervals involve two different ways of analyzing $f(x;\theta)$:

as $f(x \mid \theta)$ for Hyp. Test

and $f(\theta \mid x)$ for Conf. Int.

Hypothesis Testing:

Let $\hat{\Theta}$ be an estimator for θ. $\hat{\Theta}$ is "test statistic".

Compute $f(\hat{\Theta};\theta)$

$P(|\hat{\Theta}| > |\hat{\theta}| \mid \theta = \theta_0)$ is "p-value."

If this probability is low then we will conclude $\theta \neq \theta_0$.

Actually a bit more complicated, but this is the "idea".

Confidence Intervals

Let $\hat{\Theta}$ be an estimator for θ. "$\theta = \hat{\theta} \pm \sigma_{\hat{\theta}}$"

"standard error"

Compute $f(\hat{\Theta};\theta)$.

Find θ_A & θ_B so that

$$\left.\begin{aligned} P(\theta < \theta_A \mid \hat{\Theta} = \hat{\theta}) &= \alpha/2 \\ P(\theta > \theta_B \mid \hat{\Theta} = \hat{\theta}) &= \alpha/2 \end{aligned}\right\} \ (\theta_A, \theta_B) \text{ is "Confidence Int."}$$

(In practice, most of our examples will have $f(x;\theta)$ "rotationally symmetric" so $f(x \mid \theta) = f(\theta \mid x)$)

Discrete examples are good for checking basic understanding. Following is a "toy" discrete example.

<u>Example</u>: Suppose X has distribution depending on θ as follows.

$\boxed{\theta = 0}$

X	0	1	2
$p(x)$	$3/6$	$2/6$	$1/6$

$\boxed{\theta = 1}$

X	0	1	2
$p(x)$	$2/6$	$2/6$	$2/6$

$\boxed{\theta = 2}$

X	0	1	2
$p(x)$	$1/6$	$2/6$	$3/6$

Redo this so that it is not symmetric?

Make example have more X values so CI is more interesting?

These combine to give a joint distribution for (X, θ)
(Under assumption that $P(\theta=0) = P(\theta=1) = P(\theta=2)$)
$\hookrightarrow P(\theta=\theta_i) = 1/3$

$f(x;\theta)$	$x=0$	$x=1$	$x=2$	$f_\theta(\theta)$
$\theta=0$	$3/18$	$2/18$	$1/18$	$1/3$
$\theta=1$	$2/18$	$2/18$	$2/18$	$1/3$
$\theta=2$	$1/18$	$2/18$	$3/18$	$1/3$
$f_X(x)$	$6/18$	$6/18$	$6/18$	

<u>Hypothesis Testing</u>.

Test hypothesis "$\theta = 1$" if "$x = 2$"

<u>p-value</u> of "$x=2$" when "$\theta=1$" is

$$P(x \geq 2 \mid \theta = 1)$$

$$\hookrightarrow = \frac{P(x \geq 2 \;\&\; \theta = 1)}{P(\theta = 1)}$$

$$= \frac{2/18}{1/3} = 2/6$$

Test hypothesis "$\theta = 2$" if "$x = 1$"

<u>p-value</u> of "$x=1$" when "$\theta=2$" is

$$P(x \leq 1 \mid \theta = 2)$$

$$\hookrightarrow = \frac{P(x \leq 1 \;\&\; \theta = 2)}{P(\theta = 2)}$$

$$= \frac{1/18 + 2/18}{1/3} = 3/6$$

<u>Confidence Intervals</u>

Measure $x = 2$. Want interval for θ.

$$\left.\begin{aligned} P(\theta \leq 2 \mid x = 2) &= 1 \\ P(\theta \leq 1 \mid x = 2) &= 3/6 \\ P(\theta \leq 0 \mid x = 2) &= 1/6 \end{aligned}\right\} \quad \begin{aligned} P(\theta \in (1,2]) &= 1/2 \\ P(\theta \in (0,2]) &= 5/6 \end{aligned}$$

$\uparrow$ <u>one-sided</u> CI...